

Three-point correlators: finite-size giant magnons and singlet scalar operators on higher string levels

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Abstract

In the framework of the semiclassical approach, we compute the normalized structure constants in three-point correlation functions, when two of the vertex operators correspond to "heavy" string states, while the third vertex corresponds to a "light" state. This is done for the case when the "heavy" string states are *finite-size* giant magnons, carrying one or two angular momenta. The "light" states are taken to be singlet scalar operators on higher string levels. We first consider the case of string theory on $AdS_5 \times S^5$ dual to $\mathcal{N} = 4$ super Yang-Mills. Then we extend the obtained results to the γ -deformed $AdS_5 \times S^5_\gamma$, corresponding to $\mathcal{N} = 1$ super Yang-Mills theory, appearing as an exactly marginal deformation of $\mathcal{N} = 4$ super Yang-Mills.

1 Introduction

There have been many investigations on correlation functions in the AdS/CFT context [1]. Recently, several interesting developments have been made by considering general "heavy" string states. An efficient method to compute two-point correlation functions in the strong coupling limit is to evaluate string partition function for a "heavy" string state propagating in the AdS space between two boundary points based on a path integral approach [2, 3]. This method has been extended to the three-point functions of two "heavy" string states and a "light" mode [4, 5, 6]. Relying on these achievements, many interesting results concerning correlators of "heavy" and "light" modes have been obtained [4]-[27].

As explained in [6], there exist special massive string states vertex operators with finite quantum numbers for which the leading-order bosonic part is known explicitly and thus they can be used as candidates for "light" vertex operators in the semiclassical computation of the correlation functions. These are singlet operators which do not mix with other operators to leading nontrivial order in $\frac{1}{\sqrt{\lambda}}$ [28, 29]. An example of such scalar operator carrying no spins is [6]¹

$$V_q = (Y_4 + Y_5)^{-\Delta} \left[(\partial X_k \bar{\partial} X_k)^q + \dots \right]. \quad (1.1)$$

This operator corresponds to a scalar string state at level $n = q - 1$ so that the fermionic contributions should make the $q = 1$ state massless (BPS), with $\Delta = 4$ following from the marginality condition. The $q = 2$ choice corresponds to a scalar state on the first excited string level. In that case, we have [29]

$$\Delta(\Delta - 4) = 4(\sqrt{\lambda} - 1) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right),$$

with solution

$$\Delta = 2(\lambda^{1/4} + 1) + \frac{0}{\lambda^{1/4}} + \mathcal{O}\left(\frac{1}{\lambda^{3/4}}\right).$$

However, the subleading terms here should not be trusted as far as the fermions are expected to change the Δ -independent terms in the 1-loop anomalous dimension. For arbitrary string level n , the solution of the marginality condition with respect to Δ , to leading order in $\frac{1}{\sqrt{\lambda}}$, is given by

$$\Delta_q = 2 \left(\sqrt{(q-1)\sqrt{\lambda} + 1 - \frac{1}{2}q(q-1)} + 1 \right). \quad (1.2)$$

¹ The marginality condition for this operator is [6]: $2(1-q) + \frac{1}{2\sqrt{\lambda}} [\Delta(\Delta-4) + 2q(q-1)] + \frac{1}{(\sqrt{\lambda})^2} [\frac{2}{3}q(q-1)(q-\frac{7}{2}) + 4q] + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^3}\right) = 0$.

Let us also point out that the number q of $\partial X_k \bar{\partial} X_k$ factors in an operator never increases due to renormalization [28]. That is why, it can be used as a quantum number to characterize the leading term in the corresponding operator [30].

Here we will be interested in semiclassical computation of three-point correlation functions for the case when the "heavy" string states are *finite-size* giant magnons, carrying one or two angular momenta, while the "light" states are taken to be given by (1.1) for $q = 1, 2, 3, \dots$. First, we will consider the case of giant magnons in $AdS_5 \times S^5$. Then we will extend the obtained results to giant magnons on γ -deformed $AdS_5 \times S^5_\gamma$ background. To this end, we will use the following approach [6, 16]. The three-point functions of two "heavy" operators and a "light" operator can be approximated by a supergravity vertex operator evaluated at the "heavy" classical string configuration:

$$\langle V_H(x_1) V_H(x_2) V_L(x_3) \rangle = V_L(x_3)_{\text{classical}}.$$

For $|x_1| = |x_2| = 1$, $x_3 = 0$, the correlation function reduces to

$$\langle V_{H_1}(x_1) V_{H_2}(x_2) V_L(0) \rangle = \frac{C_{123}}{|x_1 - x_2|^{2\Delta_H}}.$$

Then, the normalized structure constants

$$\mathcal{C}_3 = \frac{C_{123}}{C_{12}}$$

can be found from

$$\mathcal{C}_3 = c_\Delta V_L(0)_{\text{classical}}, \tag{1.3}$$

where c_Δ is the normalized constant of the corresponding "light" vertex operator. Actually, we are going to compute the corresponding normalized structure constants (1.3).

2 Three-point correlators for giant magnons on $AdS_5 \times S^5$

Let us first introduce the coordinates, which we are going to use further on. If we denote the string embedding coordinates on AdS and S^5 parts of the $AdS_5 \times S^5$ background with Y and X respectively, then

$$Y_1 + iY_2 = \sinh \rho \sin \eta e^{i\varphi_1}, \quad Y_3 + iY_4 = \sinh \rho \cos \eta e^{i\varphi_2}, \quad Y_5 + iY_0 = \cosh \rho e^{it},$$

are related to the Poincare coordinates by

$$Y_m = \frac{x_m}{z}, \quad Y_4 = \frac{1}{2z} (x^m x_m + z^2 - 1), \quad Y_5 = \frac{1}{2z} (x^m x_m + z^2 + 1),$$

where $x^m x_m = -x_0^2 + x_i x_i$, with $m = 0, 1, 2, 3$ and $i = 1, 2, 3$.

For giant magnons, the AdS part of the solution, after Euclidean rotation, is given by [5, 16] ($i\tau = \tau_e$, where τ is the worldsheet time and κ is a parameter)

$$x_{0e} = \tanh(\kappa\tau_e), \quad x_i = 0, \quad z = \frac{1}{\cosh(\kappa\tau_e)}.$$

Thus, the factor $(Y_4 + Y_5)^{-\Delta}$ in (1.1) becomes $(\cosh(\kappa\tau_e))^{-\Delta}$, while $\partial X_k \bar{\partial} X_k$ is basically the string Lagrangian on the five-sphere, computed on the giant magnons' first integrals.

Since we are going to find the three-point correlators containing two heavy operators corresponding to giant magnons with one or two angular momenta, we restrict ourselves to the three-sphere. Then, we can explore the reduction of the string dynamics to the Neumann-Rosochatius integrable model by using the ansatz [31]

$$\begin{aligned} t(\tau, \sigma) &= \kappa\tau, & \theta(\tau, \sigma) &= \theta(\xi), & \phi_j(\tau, \sigma) &= \omega_j\tau + f_j(\xi), \\ \xi &= \alpha\sigma + \beta\tau, & \kappa, \omega_j, \alpha, \beta &= \text{constants}, & j &= 1, 2. \end{aligned} \quad (2.1)$$

As a consequence, the string Lagrangian in conformal gauge, on the three-sphere, can be written as (prime is used for $d/d\xi$)

$$\begin{aligned} \mathcal{L}_{S^3} &= (\alpha^2 - \beta^2) \left[\theta'^2 + \sin^2 \theta \left(f_1' - \frac{\beta\omega_1}{\alpha^2 - \beta^2} \right)^2 + \cos^2 \theta \left(f_2' - \frac{\beta\omega_2}{\alpha^2 - \beta^2} \right)^2 \right. \\ &\quad \left. - \frac{\alpha^2}{(\alpha^2 - \beta^2)^2} (\omega_1^2 \sin^2 \theta + \omega_2^2 \cos^2 \theta) \right]. \end{aligned} \quad (2.2)$$

One can prove that the first integrals of the equations of motion for $f_j(\xi)$, $\theta(\xi)$, take the form

$$\begin{aligned} f_1' &= \frac{\omega_1}{\alpha} \frac{v}{1-v^2} \left(\frac{W}{1-\chi} - 1 \right), \\ f_2' &= -\frac{\omega_1}{\alpha} \frac{uv}{1-v^2}, \\ \theta' &= \frac{\omega_1}{\alpha} \frac{\sqrt{1-u^2}}{1-v^2} \sqrt{\frac{(\chi_p - \chi)(\chi - \chi_m)}{1-\chi}}, \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} \chi_p + \chi_m &= \frac{2 - (1+v^2)W - u^2}{1-u^2}, \\ \chi_p \chi_m &= \frac{(1-W)(1-v^2W)}{1-u^2}. \end{aligned} \quad (2.4)$$

The case of *finite-size* dyonic giant magnons, corresponds to

$$0 < u < 1, \quad 0 < v < 1, \quad 0 < W < 1, \quad 0 < \chi_m < \chi < \chi_p < 1.$$

In (2.3) and (2.4) the following notations have been introduced

$$\chi = \cos^2 \theta, \quad v = -\frac{\beta}{\alpha}, \quad u = \frac{\omega_2}{\omega_1}, \quad W = \left(\frac{\kappa}{\omega_1} \right)^2.$$

The replacement into (2.2) gives (we set $\alpha = \omega_1 = 1$ for simplicity)

$$\mathcal{L}_{S^3}^{gm} = -\frac{1}{1-v^2} [2 - (1+v^2)W - 2(1-u^2)\chi]. \quad (2.5)$$

We will need also the first integral for χ , which is given by

$$\chi' = \frac{2\sqrt{1-u^2}}{1-v^2} \sqrt{\chi(\chi_p - \chi)(\chi - \chi_m)}. \quad (2.6)$$

After all that, the vertex (1.1) can be rewritten as

$$V_q = \left(\cosh(\sqrt{W}\tau_e) \right)^{-\Delta_q} (\mathcal{L}_{S^3}^{gm})^q, \quad (2.7)$$

where Δ_q should be taken from (1.2) and $\mathcal{L}_{S^3}^{gm}$ - from (2.5). Then, according to [6, 16], the normalized structure constant (1.3) becomes

$$\mathcal{C}_3^q = c_{\Delta_q} \int_{-\infty}^{\infty} \frac{d\tau_e}{\cosh^{\Delta_q}(\sqrt{W}\tau_e)} \int_{-L}^L d\sigma (\mathcal{L}_{S^3}^{gm})^q. \quad (2.8)$$

Here, the parameter L is introduced to take into account the *finite-size* of the giant magnons. Otherwise, $L \rightarrow \infty$.

The integration over τ_e in (2.8) gives

$$\int_{-\infty}^{\infty} \frac{d\tau_e}{\cosh^{\Delta_q}(\sqrt{W}\tau_e)} = \sqrt{\frac{\pi}{W}} \frac{\Gamma\left(\frac{\Delta_q}{2}\right)}{\Gamma\left(\frac{\Delta_q+1}{2}\right)},$$

where $\Gamma(x)$ is the Euler gamma-function. The integration over σ can be replaced by integration over χ according to

$$\int_{-L}^L d\sigma = 2 \int_{\chi_m}^{\chi_p} \frac{d\chi}{\chi'}, \quad (2.9)$$

where χ' is given in (2.6). Then (2.8) acquires its final form

$$\begin{aligned} \mathcal{C}_3^q &= c_{\Delta_q} \pi^{3/2} \frac{\Gamma\left(\frac{\Delta_q}{2}\right)}{\Gamma\left(\frac{\Delta_q+1}{2}\right)} \frac{(-1)^q [2 - (1+v^2)W]^q}{(1-v^2)^{q-1} \sqrt{(1-u^2)W} \chi_p} \\ &\sum_{k=0}^q \frac{q!}{k!(q-k)!} \left[-\frac{1-u^2}{1-\frac{1}{2}(1+v^2)W} \right]^k \chi_p^k {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - k; 1; 1 - \frac{\chi_m}{\chi_p}\right), \end{aligned} \quad (2.10)$$

where in accordance with (2.4)

$$\begin{aligned}\chi_p &= \frac{1}{2(1-u^2)} \left\{ q_1 + q_2 - u^2 + \sqrt{(q_1 - q_2)^2 - [2(q_1 + q_2 - 2q_1q_2) - u^2]u^2} \right\}, \\ \chi_m &= \frac{1}{2(1-u^2)} \left\{ q_1 + q_2 - u^2 - \sqrt{(q_1 - q_2)^2 - [2(q_1 + q_2 - 2q_1q_2) - u^2]u^2} \right\}, \\ q_1 &= 1 - W, \quad q_2 = 1 - v^2W,\end{aligned}\tag{2.11}$$

and ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function.

This is our general result corresponding to finite-size giant magnons with two angular momenta and to arbitrary string level $n = q - 1 = 0, 1, 2, \dots$. Now, let us give some particular examples contained in (2.10).

2.1 Giant magnons with one angular momentum

The case of finite-size giant magnons with one angular momentum $J_1 \neq 0$ corresponds to $u = 0$. This can be seen from the explicit expression for the second angular momentum J_2 :

$$\mathcal{J}_2 \equiv \frac{2\pi J_2}{\sqrt{\lambda}} = \frac{2u\sqrt{\chi_p}}{\sqrt{1-u^2}} \mathbf{E} \left(1 - \frac{\chi_m}{\chi_p} \right),$$

where $\mathbf{E}(x)$ is the complete elliptic integral of second kind. Then from (2.11) one obtains the following simplified expressions for χ_p, χ_m :

$$\chi_p = 1 - v^2W, \quad \chi_m = 1 - W.$$

Taking this into account, and using (2.10), one can find that the normalized structure constants for the first three string levels, for the case at hand, are given by:

$q = 1$ (level $n = 0$)

$$\begin{aligned}\mathcal{C}_3^1 &= 2c_{\Delta_1} \pi^{1/2} \frac{\Gamma\left(\frac{\Delta_1}{2}\right)}{\Gamma\left(\frac{\Delta_1+1}{2}\right)} \frac{1}{\sqrt{W(1-v^2W)}} \\ &\left[2(1-v^2W) \mathbf{E} \left(1 - \frac{1-W}{1-v^2W} \right) - (2 - (1+v^2)W) \mathbf{K} \left(1 - \frac{1-W}{1-v^2W} \right) \right].\end{aligned}$$

$q = 2$ (level $n = 1$)

$$\begin{aligned}\mathcal{C}_3^2 &= 2c_{\Delta_2} \pi^{1/2} \frac{\Gamma\left(\frac{\Delta_2}{2}\right)}{\Gamma\left(\frac{\Delta_2+1}{2}\right)} \frac{1}{(1-v^2)\sqrt{W(1-v^2W)}} \\ &\left[(2 - (1+v^2)W)^2 \mathbf{K} \left(1 - \frac{1-W}{1-v^2W} \right) \right. \\ &- 4(2 - (1+v^2)W)(1-v^2W) \mathbf{E} \left(1 - \frac{1-W}{1-v^2W} \right) \\ &\left. + 2\pi(1-v^2W)^2 {}_2F_1 \left(\frac{1}{2}, -\frac{3}{2}; 1; 1 - \frac{1-W}{1-v^2W} \right) \right].\end{aligned}$$

$q = 3$ (level $n = 2$)

$$\begin{aligned} \mathcal{C}_3^3 = & -2c_{\Delta_3}\pi^{1/2} \frac{\Gamma\left(\frac{\Delta_3}{2}\right)}{\Gamma\left(\frac{\Delta_3+1}{2}\right)} \frac{(2 - (1 + v^2)W)^3}{(1 - v^2)^2 \sqrt{W(1 - v^2W)}} \\ & \left[\mathbf{K}\left(1 - \frac{1 - W}{1 - v^2W}\right) - \frac{6(1 - v^2W)}{2 - (1 + v^2)W} \mathbf{E}\left(1 - \frac{1 - W}{1 - v^2W}\right) \right. \\ & + \frac{6\pi(1 - v^2W)^2}{(2 - (1 + v^2)W)^2} {}_2F_1\left(\frac{1}{2}, -\frac{3}{2}; 1; 1 - \frac{1 - W}{1 - v^2W}\right) \\ & \left. - \frac{4\pi(1 - v^2W)^3}{(2 - (1 + v^2)W)^3} {}_2F_1\left(\frac{1}{2}, -\frac{5}{2}; 1; 1 - \frac{1 - W}{1 - v^2W}\right) \right]. \end{aligned}$$

$\mathbf{K}(x)$ in the expressions above is the complete elliptic integral of first kind.

2.2 Giant magnons with two angular momenta

$q = 1$ (level $n = 0$):

$$\begin{aligned} \mathcal{C}_3^1 = & 2c_{\Delta_1}\pi^{1/2} \frac{\Gamma\left(\frac{\Delta_1}{2}\right)}{\Gamma\left(\frac{\Delta_1+1}{2}\right)} \frac{1}{\sqrt{(1 - u^2)W}\chi_p} \\ & \left[2(1 - u^2)\chi_p \mathbf{E}\left(1 - \frac{\chi_m}{\chi_p}\right) - (2 - (1 + v^2)W) \mathbf{K}\left(1 - \frac{\chi_m}{\chi_p}\right) \right]. \end{aligned}$$

$q = 2$ (level $n = 1$):

$$\begin{aligned} \mathcal{C}_3^2 = & 2c_{\Delta_2}\pi^{1/2} \frac{\Gamma\left(\frac{\Delta_2}{2}\right)}{\Gamma\left(\frac{\Delta_2+1}{2}\right)} \frac{1}{(1 - v^2)\sqrt{(1 - u^2)W}\chi_p} \\ & \left[(2 - (1 + v^2)W)^2 \mathbf{K}\left(1 - \frac{\chi_m}{\chi_p}\right) - 4(1 - u^2)(2 - (1 + v^2)W)\chi_p \mathbf{E}\left(1 - \frac{\chi_m}{\chi_p}\right) \right. \\ & \left. + 2\pi(1 - u^2)^2\chi_p^2 {}_2F_1\left(\frac{1}{2}, -\frac{3}{2}; 1; 1 - \frac{\chi_m}{\chi_p}\right) \right]. \end{aligned}$$

$q = 3$ (level $n = 2$):

$$\begin{aligned} \mathcal{C}_3^3 = & -2c_{\Delta_3}\pi^{1/2} \frac{\Gamma\left(\frac{\Delta_3}{2}\right)}{\Gamma\left(\frac{\Delta_3+1}{2}\right)} \frac{(2 - (1 + v^2)W)^3}{(1 - v^2)^2 \sqrt{(1 - u^2)W}\chi_p} \\ & \left[\mathbf{K}\left(1 - \frac{\chi_m}{\chi_p}\right) - \frac{6(1 - u^2)\chi_p}{2 - (1 + v^2)W} \mathbf{E}\left(1 - \frac{\chi_m}{\chi_p}\right) \right. \\ & + \frac{6\pi(1 - u^2)^2\chi_p^2}{(2 - (1 + v^2)W)^2} {}_2F_1\left(\frac{1}{2}, -\frac{3}{2}; 1; 1 - \frac{\chi_m}{\chi_p}\right) \\ & \left. - \frac{4\pi(1 - u^2)^3\chi_p^3}{(2 - (1 + v^2)W)^3} {}_2F_1\left(\frac{1}{2}, -\frac{5}{2}; 1; 1 - \frac{\chi_m}{\chi_p}\right) \right]. \end{aligned}$$

3 Three-point correlators for giant magnons on $AdS_5 \times S^5_\gamma$

Investigations on AdS/CFT duality [1] for the cases with reduced or without supersymmetry is of obvious interest and importance. An interesting example of such correspondence between gauge and string theory models with reduced supersymmetry is provided by an exactly marginal deformation of $\mathcal{N} = 4$ super Yang-Mills theory [32] and string theory on a β -deformed $AdS_5 \times S^5$ background suggested by Lunin and Maldacena in [33]. When $\beta \equiv \gamma$ is real, the deformed background can be obtained from $AdS_5 \times S^5$ by the so-called TsT transformation. It includes T-duality on one angle variable, a shift of another isometry variable, then a second T-duality on the first angle [34]. Taking into account that the five-sphere has three isometric coordinates, one can consider generalization of the above procedure, consisting of chain of three TsT transformations. The result is a regular three-parameter deformation of $AdS_5 \times S^5$ string background, dual to a non-supersymmetric deformation of $\mathcal{N} = 4$ super Yang-Mills [34], which is conformal in the planar limit to any order of perturbation theory [35]. The action for this γ_i -deformed ($i = 1, 2, 3$) gauge theory can be obtained from the initial one after replacement of the usual product with associative $*$ -product [33, 34, 36].

An essential property of the TsT transformation is that it preserves the classical integrability of string theory on $AdS_5 \times S^5$ [34]. The γ -dependence enters only through the *twisted* boundary conditions and the *level-matching* condition. The last one is modified since a closed string in the deformed background corresponds to an open string on $AdS_5 \times S^5$ in general.

The bosonic part of the Green-Schwarz action for strings on the γ_i -deformed $AdS_5 \times S^5_{\gamma_i}$ [37] reduced to $R_t \times S^5_{\gamma_i}$ can be written as (the common radius R of AdS_5 and $S^5_{\gamma_i}$ is set to 1)

$$\begin{aligned}
S = & -\frac{T}{2} \int d\tau d\sigma \left\{ \sqrt{-\gamma} \gamma^{ab} \left[-\partial_a t \partial_b t + \partial_a r_i \partial_b r_i + G r_i^2 \partial_a \phi_i \partial_b \phi_i \right. \right. \\
& + G r_1^2 r_2^2 r_3^2 (\tilde{\gamma}_i \partial_a \phi_i) (\tilde{\gamma}_j \partial_b \phi_j) \Big] \\
& \left. - 2G \epsilon^{ab} (\tilde{\gamma}_3 r_1^2 r_2^2 \partial_a \phi_1 \partial_b \phi_2 + \tilde{\gamma}_1 r_2^2 r_3^2 \partial_a \phi_2 \partial_b \phi_3 + \tilde{\gamma}_2 r_3^2 r_1^2 \partial_a \phi_3 \partial_b \phi_1) \right\}, \tag{3.1}
\end{aligned}$$

where T is the string tension, γ^{ab} is the worldsheet metric, ϕ_i are the three isometry angles of the deformed $S^5_{\gamma_i}$, and

$$\sum_{i=1}^3 r_i^2 = 1, \quad G^{-1} = 1 + \tilde{\gamma}_3^2 r_1^2 r_2^2 + \tilde{\gamma}_1^2 r_2^2 r_3^2 + \tilde{\gamma}_2^2 r_1^2 r_3^2. \tag{3.2}$$

The deformation parameters $\tilde{\gamma}_i$ are related to γ_i which appear in the dual gauge theory as follows

$$\tilde{\gamma}_i = 2\pi T \gamma_i = \sqrt{\lambda} \gamma_i.$$

When $\tilde{\gamma}_i = \tilde{\gamma}$ this becomes the supersymmetric background of [33], and the deformation parameter γ enters the $\mathcal{N} = 1$ SYM superpotential in the following way

$$W \propto \text{tr} \left(e^{i\pi\gamma} \Phi_1 \Phi_2 \Phi_3 - e^{-i\pi\gamma} \Phi_1 \Phi_3 \Phi_2 \right).$$

This is the case we are going to consider here.

Since the giant magnons with one or two angular momenta live on the subspace $R_t \times S_\gamma^3$ ($r_3 = 0, \phi_3 = 0$), we restrict ourselves to that subspace of $AdS_5 \times S_\gamma^5$, parameterize (see (3.2))

$$r_1 = \sin \theta, \quad r_2 = \cos \theta,$$

and use the ansatz (2.1). Then the string Lagrangian in conformal gauge, on the γ -deformed three-sphere, can be written as (prime is used for $d/d\xi$)

$$\begin{aligned} \mathcal{L} = & (\alpha^2 - \beta^2) \left[\theta'^2 + G \sin^2 \theta \left(f_1' - \frac{\beta \omega_1}{\alpha^2 - \beta^2} \right)^2 + G \cos^2 \theta \left(f_2' - \frac{\beta \omega_2}{\alpha^2 - \beta^2} \right)^2 \right. \\ & \left. - \frac{\alpha^2}{(\alpha^2 - \beta^2)^2} G (\omega_1^2 \sin^2 \theta + \omega_2^2 \cos^2 \theta) + 2\alpha\tilde{\gamma} G \sin^2 \theta \cos^2 \theta \frac{\omega_2 f_1' - \omega_1 f_2'}{\alpha^2 - \beta^2} \right], \end{aligned} \quad (3.3)$$

where

$$G = \frac{1}{1 + \tilde{\gamma}^2 \sin^2 \theta \cos^2 \theta}.$$

The equations of motion for $f_{1,2}$ following from (3.3) can be integrated once to give

$$\begin{aligned} f_1' &= \frac{1}{\alpha^2 - \beta^2} \left[\frac{C_1}{\sin^2 \theta} + \beta \omega_1 - \tilde{\gamma} (\alpha \omega_2 - \tilde{\gamma} C_1) \cos^2 \theta \right], \\ f_2' &= \frac{1}{\alpha^2 - \beta^2} \left[\frac{C_2}{\cos^2 \theta} + \beta \omega_2 + \tilde{\gamma} (\alpha \omega_1 + \tilde{\gamma} C_2) \sin^2 \theta \right], \end{aligned} \quad (3.4)$$

where $C_{1,2}$ are integration constants.

Next, we should take into account the Virasoro constraints, which for our case are given by

$$\begin{aligned} & \theta'^2 + G \sin^2 \theta \left(f_1'^2 + \frac{2\beta\omega_1}{\alpha^2 + \beta^2} f_1' + \frac{\omega_1^2}{\alpha^2 + \beta^2} \right) \\ & + G \cos^2 \theta \left(f_2'^2 + \frac{2\beta\omega_2}{\alpha^2 + \beta^2} f_2' + \frac{\omega_2^2}{\alpha^2 + \beta^2} \right) = \frac{\kappa^2}{\alpha^2 + \beta^2}, \\ & \theta'^2 + G \sin^2 \theta \left(f_1'^2 + \frac{\omega_1}{\beta} f_1' \right) + G \cos^2 \theta \left(f_2'^2 + \frac{\omega_2}{\beta} f_2' \right) = 0. \end{aligned} \quad (3.5)$$

Replacing (3.4) in the constraints (3.5) one finds the first integral θ' of the equation of motion for θ and a relation among the parameters

$$\theta'^2 = \frac{1}{(\alpha^2 - \beta^2)^2} \left[(\alpha^2 + \beta^2)\kappa^2 - \frac{C_1^2}{\sin^2 \theta} - \frac{C_2^2}{\cos^2 \theta} \right. \quad (3.6)$$

$$\left. - (\alpha\omega_1 + \tilde{\gamma}C_2)^2 \sin^2 \theta - (\alpha\omega_2 - \tilde{\gamma}C_1)^2 \cos^2 \theta \right],$$

$$\omega_1 C_1 + \omega_2 C_2 + \beta\kappa^2 = 0. \quad (3.7)$$

Now, we introduce the variable

$$\chi = \cos^2 \theta,$$

and the parameters

$$v = -\frac{\beta}{\alpha}, \quad u = \frac{\Omega_2}{\Omega_1}, \quad W = \left(\frac{\kappa}{\Omega_1} \right)^2, \quad K = \frac{C_2}{\alpha\Omega_1},$$

$$\Omega_1 = \omega_1 \left(1 + \tilde{\gamma} \frac{C_2}{\alpha\omega_1} \right), \quad \Omega_2 = \omega_2 \left(1 - \tilde{\gamma} \frac{C_1}{\alpha\omega_2} \right).$$

By using them and (3.7), the three first integrals can be rewritten as

$$f'_1 = \frac{\Omega_1}{\alpha} \frac{1}{1-v^2} \left[\frac{vW - uK}{1-\chi} - v(1 - \tilde{\gamma}K) - \tilde{\gamma}u\chi \right],$$

$$f'_2 = \frac{\Omega_1}{\alpha} \frac{1}{1-v^2} \left[\frac{K}{\chi} - uv(1 - \tilde{\gamma}K) - \tilde{\gamma}v^2W + \tilde{\gamma}(1 - \chi) \right], \quad (3.8)$$

$$\theta' = \frac{\Omega_1}{\alpha} \frac{\sqrt{1-u^2}}{1-v^2} \sqrt{\frac{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}{\chi(1-\chi)}},$$

where

$$\chi_p + \chi_m + \chi_n = \frac{2 - (1 + v^2)W - u^2}{1 - u^2},$$

$$\chi_p \chi_m + \chi_p \chi_n + \chi_m \chi_n = \frac{1 - (1 + v^2)W + (vW - uK)^2 - K^2}{1 - u^2}, \quad (3.9)$$

$$\chi_p \chi_m \chi_n = -\frac{K^2}{1 - u^2}.$$

The case of dyonic finite-size giant magnons we are interested in, corresponds to

$$0 < u < 1, \quad 0 < v < 1, \quad 0 < W < 1, \quad 0 < \chi_m < \chi < \chi_p < 1, \quad \chi_n < 0.$$

Replacing (3.8) and (3.9) in (3.3), one can find the final form of the Lagrangian to be (we set $\alpha = \Omega_1 = 1$ for simplicity)

$$\mathcal{L}_\gamma = -\frac{1}{1-v^2} \left[2 - (1 + v^2)W - 2\tilde{\gamma}K - 2(1 - \tilde{\gamma}K - u(u - \tilde{\gamma}uK + \tilde{\gamma}vW))\chi \right] \quad (3.10)$$

After setting $\tilde{\gamma} = 0$ in (3.10) it coincides with (2.5) as it should be. Now, in the γ -deformed case, the first integral for χ reads

$$\chi' = \frac{2\sqrt{1-u^2}}{1-v^2} \sqrt{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}. \quad (3.11)$$

To compute the normalized structure constant (1.3) for the case of two finite-size dyonic giant magnons living on the γ -deformed three-sphere S_γ^3 , we have to modify (2.8) by using (3.10)

$$\mathcal{C}_3^q \rightarrow \mathcal{C}_{3\gamma}^q = c_{\Delta_q} \int_{-\infty}^{\infty} \frac{d\tau_e}{\cosh^{\Delta_q}(\sqrt{W}\tau_e)} \int_{-L}^L d\sigma (\mathcal{L}_\gamma)^q. \quad (3.12)$$

Replacing the integration over σ according to (2.9) and using (3.11), one finds

$$\begin{aligned} \mathcal{C}_{3\gamma}^q &= c_{\Delta_q} \pi^{3/2} \frac{\Gamma\left(\frac{\Delta_q}{2}\right)}{\Gamma\left(\frac{\Delta_q+1}{2}\right)} \frac{(-2A)^q}{(1-v^2)^{q-1} \sqrt{(1-u^2)W(\chi_p - \chi_n)}} \\ &\sum_{k=0}^q \frac{q!}{k!(q-k)!} \left(-\frac{B}{A}\right)^k \chi_p^k F_1\left(\frac{1}{2}, \frac{1}{2}, -k; 1; 1-\epsilon, 1-\frac{\chi_m}{\chi_p}\right), \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} A &= 1 - \frac{1}{2}(1+v^2)W - \tilde{\gamma}K, \quad B = 1 - \tilde{\gamma}K - u[u - \tilde{\gamma}(Ku - vW)], \\ \epsilon &= \frac{\chi_m - \chi_n}{\chi_p - \chi_n}. \end{aligned} \quad (3.14)$$

Here, $F_1(a, b_1, b_2; c; z_1, z_2)$ is one of the hypergeometric functions of two variables (*Appell* F_1). In writing (3.13), we used the following property of $F_1(a, b_1, b_2; c; z_1, z_2)$ [38]

$$F_1(a, b_1, b_2; c; z_1, z_2) = (1-z_1)^{-b_1} (1-z_2)^{-b_2} F_1\left(c-a, b_1, b_2; c; \frac{z_1}{z_1-1}, \frac{z_2}{z_2-1}\right).$$

Then, the arguments of F_1 , $(1-\epsilon, 1-\chi_m/\chi_p)$ belong to the interval $(0,1)$. This representation gives the possibility to use the defining series for $F_1(a, b_1, b_2; c; z_1, z_2)$,

$$F_1(a, b_1, b_2; c; z_1, z_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(a)_{k_1+k_2} (b_1)_{k_1} (b_2)_{k_2}}{(c)_{k_1+k_2}} \frac{z_1^{k_1} z_2^{k_2}}{k_1! k_2!}, \quad |z_1| < 1, \quad |z_2| < 1,$$

in order to consider the limits $\epsilon \rightarrow 0$, $\chi_m/\chi_p \rightarrow 0$, or both. The small ϵ limit corresponds to taking into account the *leading* finite-size effect, while $\epsilon = 0$, $\chi_m = 0$, $\chi_n = 0$, $K = 0$, $W = 1$, describes the infinite-size case.

Now, let us write down what the general formula (3.13) for the normalized structure constant in the γ -deformed case gives for the first two string levels.

$q = 1$ (level $n = 0$):

$$\begin{aligned} \mathcal{C}_{3\gamma}^1 &= 2c_{\Delta_1}\pi^{3/2} \frac{\Gamma\left(\frac{\Delta_1}{2}\right)}{\Gamma\left(\frac{\Delta_1+1}{2}\right)} \frac{1 - \frac{1}{2}(1+v^2)W - \tilde{\gamma}K}{\sqrt{(1-u^2)W(\chi_p - \chi_n)}} \\ &\left[\frac{1 - u^2 - \tilde{\gamma}(uvW + (1-u^2)K)}{1 - \frac{1}{2}(1+v^2)W - \tilde{\gamma}K} \chi_p F_1\left(1/2, 1/2, -1; 1; 1 - \epsilon, 1 - \chi_m/\chi_p\right) \right. \\ &\left. - \frac{2}{\pi} \mathbf{K}(1 - \epsilon) \right]. \end{aligned}$$

$q = 2$ (level $n = 1$):

$$\begin{aligned} \mathcal{C}_{3\gamma}^2 &= 4c_{\Delta_2}\pi^{3/2} \frac{\Gamma\left(\frac{\Delta_2}{2}\right)}{\Gamma\left(\frac{\Delta_2+1}{2}\right)} \frac{(1 - \frac{1}{2}(1+v^2)W - \tilde{\gamma}K)^2}{(1-v^2)\sqrt{(1-u^2)W(\chi_p - \chi_n)}} \\ &\left[\frac{2}{\pi} \mathbf{K}(1 - \epsilon) - 2 \frac{1 - u^2 - \tilde{\gamma}(uvW + (1-u^2)K)}{1 - \frac{1}{2}(1+v^2)W - \tilde{\gamma}K} \chi_p F_1\left(1/2, 1/2, -1; 1; 1 - \epsilon, 1 - \chi_m/\chi_p\right) \right. \\ &\left. + \left(\frac{1 - u^2 - \tilde{\gamma}(uvW + (1-u^2)K)}{1 - \frac{1}{2}(1+v^2)W - \tilde{\gamma}K} \right)^2 \chi_p^2 F_1\left(1/2, 1/2, -2; 1; 1 - \epsilon, 1 - \chi_m/\chi_p\right) \right]. \end{aligned}$$

Let us point out that in the γ -deformed case, we can not obtain the reduction of (3.13) to the giant magnons with one nonzero angular momentum just by setting $u = 0$. It follows from the observation that now the smallest consistent reduction is to the $R_t \times S_\gamma^3$ subspace of $AdS_5 \times S_\gamma^5$ due to the *twisted* boundary conditions [39]. This becomes clear if one looks at the explicit expression for the second angular momentum

$$\mathcal{J}_2 \equiv \frac{2\pi J_2}{\sqrt{\lambda}} = \frac{2}{\sqrt{1-u^2}} \left[\frac{u\chi_n - vK}{\sqrt{\chi_p - \chi_n}} \mathbf{K}(1 - \epsilon) + u\sqrt{\chi_p - \chi_n} \mathbf{E}(1 - \epsilon) \right].$$

4 Concluding Remarks

Recently, some progress have been made in computing the *finite-size* effects on the three-point correlation functions in AdS/CFT context, in the framework of the semiclassical approximation [19, 20, 24, 26]. This was done for the case when the "heavy" string states are finite-size giant magnons, carrying one or two angular momenta, while the "light" state was taken to be represented by the dilaton operator. The case of "light" primary scalar operator was also investigated in [26].

Here, we considered the case, when the "light" states are taken to be singlet scalar operators on *higher* string levels [6]. We first considered the case of string theory on $AdS_5 \times S^5$ dual to $\mathcal{N} = 4$ super Yang-Mills. Then we extended the obtained results to the γ -deformed

$AdS_5 \times S^5_\gamma$ background, corresponding to $\mathcal{N} = 1$ super Yang-Mills theory, arising as marginal deformation of $\mathcal{N} = 4$ super Yang-Mills. We hope that our contribution will help for better understanding of the AdS/CFT duality at the level of holographic correlation functions.

There are other explicitly known vertexes (see [6] and references therein), so it will be an interesting task to consider the finite-size effects in these cases also.

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